Ordinary Differential Equations
Lectures notes

1 What is this course

The subject of differential equations can be described as the study of equations involving derivatives. It can also be described as the study of anything that changes. The reason for this goes back to differential calculus, where one learns that the derivative of a function describes the rate of change of the function. Thus any quantity that varies can be described by an equation involving its derivative, whether the quantity is a position, velocity, temperature, population or volume.

There are three main ways to study differential equations. There are analytic methods, wherein a mathematical formula for a solution of a differential equation is obtained. There are Numerical techniques, which provide an approximate solution, generally using a computer or programmable calculator. Differential Equations can also be studied qualitatively, determining general properties of solution without concern for exact behavior.

In this course, we will emphasize analytic methods, but Qualitative and numerical techniques will be left to other courses.

2 Introduction

An ordinary differential equation (ODE) is an equation involving an unknown function of one variable and some its derivatives, while a partial differntial equation (PDE) can be defined as is an equation involving an unkown function of two or more variables and certian of its partial derivatives.

Examples

1- The equation

$$\frac{du}{dt} = y^2, \tag{2.1}$$

where $u : R \longrightarrow R$, is an ODE .

2- The equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 y}{\partial x^2},$$

where $u : R^2 \longrightarrow R$, is a PDE.

Remark 2.1. in the ODEs we may refer for simplisity $\frac{dy}{dt} = y_t$ or $y^{'}$, therefore equation (2.1) can be rewrittin in this way

$$y^{'} = y^2.$$

Definition 2.2. The order of any differential equation is the order highest derivative which appears in the equation.

Definition 2.3. For any diffferential equation, we say that it is linear when it is linear with respect to the dependent variable y, otherwise we say that the equation is nonlinear.

Examples

1-

$$y^{''} + y^{'} + y = \sin x, \qquad \text{is a second order linear ODE.}$$

2-

$$y^{'} + y^2 = 0, \qquad \text{is a first order nonlinear ODE}$$

3-

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \qquad \text{is a second order linear PDE}$$

Definition 2.4. The function $y = y(t)$, is called is a solution to a ODE on the open interval I, if it satisfies the equation and defined on I.

Example

it is easy to see that the function

$$y = \frac{1}{(c-t)}, \tag{2.2}$$

is defind on $R/\{c\}$, where $c \in R$,

and satisfy of the following ODE

$$y^{'} = y^2. \tag{2.3}$$

Therefore, it is a solution to this ODE on $R/\{c\}$.

Definition 2.5. the problem of an ODE with the intial condition $y(t_0) = y_0$, is called intial value problem (IVP).

Example consider the IVP of equation (2.3), with the intial value condition $y(0) = 1$, we see that $y(0) = \frac{1}{(c)} = 1$, thus $c = 1$. Therefore, the solution of this IVP takes the form

$$y = \frac{1}{(1-t)}.$$

Exercises

1- For each of the following differential equations study the type (ODE or PDE), (Linear or nonlinear), and show the order.

(i) $y^{'} = \sin(y) + t$

(ii) $y_t = y_x + e^{t+x}$

(iii) $\cos(y^{''}) = t^2$

(iv) $y^{''} + y = \tan(t)$.

2- show that the function $y = ke^t$, where k is a constant, is the solution of the following ODE

$$\frac{dy}{dt} = y,$$

and then study the solution of the IVP of this equation with the intial condition

$$y(0) = 2.$$

3 Methods for Solving First Order Equations

We will study some mothds used to find the solutions of the first order equations which take the form

$$y^{'} = f(y,t).$$

1- Separable Equations

Finding a way to separate the variables is almost always the best method to attempt first when trying to solve a differential equation. Even if one of the methods that we will discuss later works for a given differential equation, we will invariably end up with the same integral to solve. Formally, a differential equation is separable if it can be written in the form

$$\frac{dy}{dt} = f(y,t) = a(t)b(y)$$

where $a, b : R \longrightarrow R$ are continuous functions
and the solution is

$$\int \frac{dy}{b(y)} = \int a(t)dt.$$

It is not always easy to determine whether or not a given differential equation is separable. The following theorem addresses this problem.

Theorem 3.1. *The differential equation $y' = f(y,t)$, is separable if and only if*

$$f(t,y)\frac{\partial^2 f}{\partial t \partial y} = \frac{\partial f}{\partial t}\frac{\partial f}{\partial y}.$$

Example
Determine if $y' = 1 + t^2 + y^3 + t^2y^3$, is separable

Setting $f(t,y) = 1 + t^2 + y^3 + t^2y^3$ and taking the necessary partial derivatives,

$$\frac{\partial f}{\partial t} = 2t + 2ty^3,$$

$$\frac{\partial f}{\partial y} = 3y^2 + 3t^2y^2,$$

Hence

$$\frac{\partial f}{\partial t}\frac{\partial f}{\partial y} = 6ty^2 + 6t^3y^2 + 6ty^5 + 6t^3y^5.$$

and

$$f(t,y)\frac{\partial^2 f}{\partial t \partial y} = 6ty^2 + 6t^3y^2 + 6ty^5 + 6t^3y^5 = \frac{\partial f}{\partial t}\frac{\partial f}{\partial y}$$

Consequently, the differential equation is separable.
Exercise For the equation in the last example, find a formula for the solution.

Example
Find the solution of the following equation

$$y' = y\sin(t).$$

It is clear that this equation is separable, and it can be written as following

$$\frac{dy}{y} = \sin(t)dt$$

integrate the two sides, it follows that

$$\ln(y) = -\cos(t) + c,$$

thus

$$y = e^{-\cos(t)}e^{c}.$$

set $k = e^c$, we get

$$y = ke^{-\cos(t)}.$$

2- Homogeneous Equations

An ordinary differential equation is said to be a homogeneous differential equation if the following condition is satisfied

$$y^{'} = f(zt, zy) = f(t, y),$$

for any $z \in R$.

Set $y = vt$, thus the general form of first order ODE becomes

$$y^{'} = \frac{dy}{dt} = \frac{dt(vt)}{dt} = v + t\frac{dv}{dt} = f(t, vt).$$

Since this equation is homogenous, we can use separation of variables to solve the equation

$$v^{'} = \frac{f(t, vt) - v}{t}.$$

Example

Find the solution of the following equation

$$y^{'} = \frac{y^2 + 2ty}{t^2}.$$

set

$$f(y, t) = \frac{y^2 + 2ty}{t^2}$$

Clearly,

$$f(zt, zy) = \frac{(zy)^2 + 2(zt)(zy)}{(zt)^2} = f(t, y).$$

Therefore, this equation is homogenuos
Now to find the solution, we set $y = vt$, and the equation can be written as follows

$$v^{'} = \frac{\frac{v^2t^2+2tvt}{t^2} - v}{t} = \frac{v^2 - v}{t}$$

Thus

$$\frac{dt}{t} = \frac{dv}{v^2 + v}$$

if you integrate the two sides, we get

$$\ln(t) = \int \frac{dv}{v(v+1)} = \int (\frac{A}{v} + \frac{B}{v+1})dt$$

It is not difficult to see that $A = 1, B = -1$, thus the last equation becomes

$$\ln(t) = \ln(v) - \ln(v+1) + c = \ln(\frac{v}{v+1}) + c.$$

Thus

$$t = \frac{v}{(v+1)}e^c,$$

set $e^c = k$, we get

$$t = k\frac{v}{v+1}.$$

Thus

$$tv + t = kv.$$

i.e. $v(t-k) = -t$, thus

$$v = \frac{y}{t} = \frac{t}{k-t}$$

Therefore,

$$y = \frac{t^2}{(k-t)}.$$

as the general solution of the original differential equation.

Exercise Find the general solution of $y^{'} = (y/t) - 1$.

3- Exact Equations
Consider the differential equation which takes the form

$$M(t, y)dt + N(t, y)dy = 0,$$

we say that this differntial equation is exact if it catisfied this condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

To solve an Exact Equation $M(t, y)dt+N(t, y)dy = 0$, we have to follow the following steps

(i) Assume that the function ϕ is a for t and y (the solution of the general equation), such that

(ii) Set $M(t, y) = \frac{\partial \phi}{\partial t}$, $N(t, y) = \frac{\partial \phi}{\partial y}$

(iii) Integrate $M(t, y) = M(t, y) = \frac{\partial \phi}{\partial t}$ in ti t to optain

$$\phi(t, y) = \int_t M(s, y)ds + h(y)$$

(iv) Calculate $\frac{\partial \phi}{\partial y}$ from the expression for $\phi(t, y)$ in step 2. The solution is $\phi(t, y) = C$, where C is a constant.

(v) Set the expression for $\frac{\partial \phi}{\partial y}$ obtained in step (3) equal to $N(t, y)$. This should give a differential equation for $h(y)$.

(vi) Solve for $h(y)$.

(vii) Substitute the expression for $h(y)$ into the expression for $\phi(t, y)$ in step (2). The solution is $\phi(t, y) = C$, where C is aconstant.

Example Find the solution of the following differential equation

$$y^{'} = -\frac{y\cos(t) + 2te^y}{\sin(t) + t^2e^y + 2}.$$

We can rewrite the differential equation as

$$(y\cos(t) + 2te^y)dt = (\sin(t) + t^2e^y + 2)dy$$

which has the form $M(t, y)dt + N(t, y)dy = 0$, where

$$M(t, y) = (y\cos(t) + 2te^y), \quad N(t, y) = (\sin(t) + t^2e^y + 2).$$

It is clear that

$$\frac{\partial M}{\partial y} = \cos(t) + 2te^y = \frac{\partial N}{\partial t}.$$

Assume that the function ϕ is a for t and the solution y of the general equation such that

$$\frac{\partial \phi}{\partial t} = M(t, y) = y\cot(t) + 2te^{y}, \tag{3.1}$$

$$\frac{\partial \phi}{\partial y} = N(t, y) = \sin(t) + t^2e^{y} + 2. \tag{3.2}$$

Integrate equation (3.1) over t, it follows that

$$\phi(t, y) = \int (y\cot(t) + 2te^{y})dt = y\sin(t) + t^2e^{y} + h(y), \tag{3.3}$$

where h is an unknown function of y.

Differentiating the last equation with respect to y and setting the result equal to (3.2) gives

$$\frac{\partial \phi}{\partial y} = \sin(t) + t^2e^{y} + 2 = \sin(t) + t^2e^{y} + h^{'}(y),$$

Canceling common terms of both sides of the equation gives $h^{'}(y) = 2$ or $dh = 2dy$, which leads to

$$h(y) = 2y + c$$

Thus equation (3.3) becomes

$$\phi(t, y) = y\sin(t) + t^2e^{y} + 2y + c,$$

Therefore, if we consider $c = 0$, the family for the solution of the general equation takes the form

$$y\sin(t) + t^2e^{y} + 2y = C,$$

where C is a constant.

Exercise For the equation in the last example, study the solution of the IVP , where $y(0) = 2$.

Exercise Solve $(t + 2y)dt + (2t - y)dy = 0$

4-Integrating Factors

We study now the linear equation, which takes the form

$$y^{'} = a(t)y + b(t). \tag{3.4}$$

To find the solution for this type of equation, we need to follow thr following steps

(i) rewrite the equation in the form
$dy = a(t)ydt + b(t)dt.$

(ii) Set $\mu = e^{-\int a(t)dt}$.

(iii) multiblate the equation in step 1 by μ, to get

$$e^{-\int a(t)dt}dy = a(t)e^{-\int a(t)dt}dty + b(t)e^{-\int a(t)dt}dt.$$

(iv) we write the equation in step 3 in the form

$$d(e^{-\int a(t)dt}y) = b(t)e^{-\int a(t)dt}dt.$$

(v) integrate the two sides of the equation in last step, we get

$$y = e^{\int a(t)dt} \int b(t)e^{-\int a(t)dt}dt.$$

Example
Find the solution of the following IVP

$$t^3y' = t^2y + 5, \qquad y(1) = 1.$$

Firstly, we need to find the solution of the differntial equation

now, we rewrite the equation in the form of (3.4),

$$\frac{dy}{dt} = \frac{y}{t} + \frac{5}{t^3}. \tag{3.5}$$

set

$$\mu = e^{-\int \frac{dt}{t}} = e^{-\ln(t)} = \frac{1}{t}.$$

multiblate equation (3.5) by μ, we get

$$\frac{1}{t}dy - \frac{dt}{t^2}y = \frac{5}{t^4}dt.$$

Thus,

$$d(\frac{y}{t}) = \frac{5}{t^4}dt.$$

integrate the two sides

$$\frac{y}{t} = \frac{-25}{t^5} + c,$$

Thus,

$$1 = y(1)/1 = -25/1 + c,$$

which means $c = 26$, therefore, the solution of the IVP takes the form

$$y = \frac{-25}{t^4} + 26t.$$

Bernoulli Equations

These equations are similar in form to equation (3.4), although they are not linear, and have the form

$$y' = a(t)y + y^n b(t), \qquad n \in Z, n \neq 0, 1. \tag{3.6}$$

Bernoulli equations can be made linear by making the substitution

$$z = y^{1-n},$$

Differentiating,

$$\frac{dz}{dt} = (1-n)y^{-n}y'.$$

Thus

$$y' = \frac{y^n}{(1-n)}z'.$$

Substituting the last equation in (reftv), gives

$$\frac{y^n}{(1-n)}z' = a(t)y + y^n b(t),$$

Thus

$$z' = a(t)(1-n)z + (1-n)b(t).$$

which a linear equation for z can be solved by using **Integrating Factors** method.

Example

Solve the IBP

$$y' + ty = \frac{t}{y^3}, \qquad y(1) = 2.$$

This is a Bernoulli equation with $n = -3$, so we let

$$z = y^{1-n} = y^4.$$

Thus $z^{'} = 4y^3y^{'}$, which leads to $y^{'} = z^{'}/(4y^3)$.

Therefore, the oreginal differential equation becames

$$\frac{z^{'}}{4y^3} + ty = \frac{t}{y^3}.$$

Thus

$$z^{'} + 4tz = 4t, \quad z(1) = y(1)^4 = 2^4 = 16.$$

The integrating factor is $\mu = e^{\int 4tdt} = e^{2t^2}$.

Multiplying the equation by μ, it follows that

$$e^{2t^2}dz + 4te^{2t^2}z = 4te^{2t^2}.$$

Thus

$$d(e^{2t^2}z) = 4te^{2t^2}dt.$$

Integrate the two sides to get

$$e^{2t^2}z = e^{2t^2} + c,$$

Thus

$$z = 1 + \frac{c}{e^{2t^2}}$$

Therefore,

$$z(1) = 16 = 1 + \frac{c}{e^2}.$$

i.e.

$$c = 15e^2,$$

so that

$$z = 1 + \frac{15e^2}{e^{2t^2}}.$$

It follows that

$$y = z^{\frac{1}{4}} = (1 + \frac{15e^2}{e^{2t^2}})^{\frac{1}{4}}.$$

Exercises

1- Determine if either of the following equations are separable.

$$y' = \cos(t+y) + \cos(t-y), \qquad y' = \cos(t+y) + \sin(t-y).$$

2- Solve the following homogeneous equations

(i) $y' = \frac{y}{t}(\frac{y}{t} + 1)$,

(ii) $y' = \frac{t^2-3y^2}{ty}$,

(iii) $y' = \frac{3t-4y}{3t+4y}$,

(iv) $ty' = y + \tan(\frac{y}{t})$,

(v) $y' = \frac{y}{t}[\frac{1}{\ln(\frac{y}{t})} - 1]$.

3- Determine which of the following differential equations are exact

$$(t^2 + ty)dt + tydy = 0,$$
$$(2y + y^2)dt - tdy = 0,$$
$$t^2y^3dt + t^3y^2dy = 0,$$
$$(e^t + y)dt + (2y + t + ye^y)dy = 0.$$

4- Show that the following equations are exact and then solve them

$$\frac{xdx}{(x^2+y^2)^{3/2}} + \frac{ydy}{(x^2+y^2)^{3/2}},$$

$$\frac{dy}{dx} = \frac{y + 6x^2}{x(2 - \ln(x))},$$

$$tdt + ydy = 0.$$

, 5- Solve the IVP,

$$y' + y = \cos t, \qquad y(0) = 1.$$

$$ty' + 2y = e^t, \qquad t > 0,$$

$$y' - \frac{3}{t}y = 4y^{-5}, \qquad y(1) = 2.$$

4 Methods for Solving Higher Order Linear Equations with Constant Coefficients

The general second order linear equation with constant coefficients is

$$ay'' + by' + cy = f(t), \tag{4.1}$$

where a, b, and c are constants, $a \neq 0$, f is a function of t, if $f = 0$, then we say that the equaton (4.1) is *homogeneous*,

$$ay'' + by' + cy = 0, \tag{4.2}$$

other wise it is called *inhomogeneous* equation.

The second order linear initial value problem with constant coefficients is then

$$ay'' + by' + cy = f(t), \qquad y(t_0) = y_0,\ y'(t_0) = y_1. \tag{4.3}$$

Equation (4.3) is sometimes called the *Cauchy initial value problem.* We refer to the solution of equation (4.1), which involves two arbitrary constants as the general solution of the differential equation, to distinguish it from the solution of the initial value problem (8.6), which has no arbitrary constants.

Instead of constraining the solution and its derivative at the same point t_0 , it is also possible to specify that value of the solution at two points as in the *Dirichlet Problem* or boundary value problem (BVP) with Dirichlet conditions,

$$ay'' + by' + cy = f(t), \qquad y(t_0) = y_0,\ y(t_1) = y_1,$$

or to specify the value of the derivative of the solution at two points as in the *Neumann Problem*, or boundary value problem with Neuman boundary conditions,

$$ay'' + by' + cy = f(t), \qquad y'(t_0) = y_0,\ y'(t_1) = y_1.$$

Also we can consider the solution and its derivative at two different points, as in the *initial-boundary value problem*,

$$ay'' + by' + cy = f(t), \qquad a_{11}y(t_0) + a_{12}y'(t_0) = y_0,\ a_{21}y(t_1) + a_{22}y'(t_1) = y_1.$$

where a_{11}, a_{12}, a_{21} and a_{22} are constants. We will only study the Cauchy initial value problem, (4.3) and will not consider any of the boundary value problems, which are considerably more complicated.

We use the notation $y_H(t)$ to denote the general solution of (4.2), and $y_P(t)$ to denote any particular solution of (4.1) that is not a solution of (4.2). We will see that there may be many different solutions to the homogeneous equation (8.10). In fact every solution of the homogeneous equation can be written in the form

$$y = c_1 y_1(t) + c_2 y_2(t), \tag{4.4}$$

where $y_1(t)$ and $y_2(t)$ are each by themselves solutions of (4.2), then we call (4.4) the general solution of the homogeneous equation, and call the pair of functions $\{y_1(t), y_2(t)\}$ and fundamental set of solutions to the homogeneous equation. Then the general solution of (4.1) is

$$y(t) = y_H(t) + y_P(t).$$

The functions $y_1(t)$ and $y_2(t)$ must not only both be solutions of (4.2), but they must both be linearly independent (in the sense that it is impossible to find any pair of constants c_1 and c_2, both non-zero such that $c_1 y_1(t) + c_2 y_2(t) = 0, \quad \text{for all } t$).

Example: Show that the two solutions of the homogeneous linear differential equation

$$y^{"} - 7y^{'} + 12y = 0,$$

are $y_1(t) = e^{4t}$ and $y_2(t) = e^{3t}$, and then show that $y = c_1 y_1 + c_2 y_2$, is the general solution of this equation.

Clearly, $y_1^{'} = 3e^{3t}$, $y_1^{''} = 9e^{3t}$, and $y_1^{"} + 7y_1^{'} + 12y_1 = e^{3t}[9 - 7(3) + 12] = 0$,

$y_2^{'} = 4e^{4t}$, $y_2^{''} = 16e^{4t}$, and $y_2^{"} + 7y_2^{'} + 12y_2 = e^{4t}[16 - 7(4) + 12] = 0$.

and they are linearly independent, because if $c_1 e^{3t} + c_2 e^{4t} = 0$, then it should be both of $c_1 = c_2 = 0$.

Morever,

$$y^{"} + 7y^{'} + 12y = c_1[y_1^{"} + 7y_1^{'} + 12y_1] + c_2[y_2^{"} + 7y_2^{'} + 12y_2] = c_1(0) + c_2(0) = 0.$$

Examle: Find the general solution of the homogeneous linear equation

$$y^{''} + 6y^{'} = 0.$$

Let $z = y^{'}$, yields $z^{'} + 6z = 0$, thus by separation of variables, we have

$$\int \frac{dz}{z} = -\int 6dt,$$

Thus

$\ln z = -6t + c_1$, which leads to $y^{'} = e^{-6t}e^{c_1}$.
Thus

$$dy = Ce^{-6t}dt, \qquad \text{where} \quad C = e^{c_1}.$$

integrate again the last equation, it follows that

$$y = -\frac{C}{6}e^{-6t} + C_2,$$

or

$$y = C_1e^{-6t} + C_2, \qquad \text{where} \quad C_1 = -\frac{C}{6}.$$

Example Find the general solution of $y^{''} + 6y^{'} = t$.
Assuming that $z = y^{'}$, then the equation becomes

$$z^{'} + 6z = t.$$

Since the last equation is linear in z, we can solve this equation using the integrating facter and we can get that the general solution is

$$y = (1/12)t^2 - (1/36)t + C1e^{-6t} + C_2.$$

We may think about a way (different from the way which has been used in the last two examples), to find the solution of the equation

$$ay^{''} + ay^{'} + cy = f(t), \tag{4.5}$$

So, we need to follow these steps:

(i) Firstly, we find the homogeneous solution, y_H, we assume that $y = e^{\lambda t}$. We get two values λ_1, λ_2, such that

$$y_H = C_1e^{\lambda_1} + C_2e^{\lambda_2}.$$

(ii) Secondly, we find the inhomogeneous solution, y_p,we assume that y has the same shape of f (y_p quadratic polynomial in case of f is a polynomial or y_p is a circular function in case of f is a circular function).

(iii) The general solution of the origenral equation (4.5), is $y = y_H + y_p$.

Example: For the last example

$$y'' + 6y' = t, \tag{4.6}$$

find y_H, and y_p, and then what is the general solution?.

Assume that $y_H = y = e^{\lambda t}$, which leads to $y' = \lambda e^{\lambda t}$, $y'' = \lambda^2 e^{\lambda t}$.

Thus the homogeneous equation becomes

$$y'' + 6y' = e^{\lambda t}[\lambda^2 + 6\lambda] = 0.$$

Since $e^{\lambda t} \neq 0$, then $\lambda(\lambda + 6) = 0$, thus $\lambda = 0$ or $\lambda = -6$. Therefore,

$$y_H = C_1 e^{0t} + C_2 e^{6t} = C_1 + C_2 e^{6t}.$$

Next, we assume that $y_p = y = at^2 + bt + c$, $y' = 2at + b$, and $y'' = 2a$.
Thus

$$y'' + 6y' = 2a + 6[2at + b] = 2a + 12at + 6b = t,$$

So

$$2a + 6b = 0, \qquad 12a = 1,$$

Thus $a = 1/12$, $b = (-1/3)$, $a = (-1/36)$.
Therefore, $y_p = (1/12)t^2 - (1/3)t + c$, assume that $c = 0$.
Thus

$$y = y_H + y_p = C_1 + C_2 e^{6t} + (1/12)t^2 - (1/36)t.$$

Which is the general solution of (4.6).